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MECHANICS OF DEFORMATION OF POROUS ELASTIC MEDIA WITH MICRO-STRUCTURE

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Abstract—A unified treatment of the mechanics of deformation of a nondissipative liquid filled saturated porous elastic medium with microstructure is presented. The strain energy of a porous elastic continuum is introduced and a method due to R. D. Mindlin is employed to formulate the first gradient theory of porous elastic continuum. A linear theory is developed and the basic dynamical equations for infinitesimal motion of fluid–solid system are obtained and discussed. Wave propagation is considered and the existance of five elastic body waves, two shear waves and three dilatational waves are established.

1. INTRODUCTION

The linearized couple-stress theories in which the couple and double stresses are taken into account have become an active field of research in recent years. The modern derivation of the couple stress theories has been given by Truesdell and Toupin[1], Toupin[2] and Mindlin and Tiersten[3]. A more general theory taking into account all terms of the gradient of the strain tensor in contrast to the previous works where only the gradient of curl of displacement were considered was obtained by Mindlin[4]. In a subsequent paper Mindlin and Eshel[5], gave a derivation of the basic equations based on conservation principles. A general thermodynamical treatment of strain gradient theories was given by Green and Rivlin[6]. Recently, Ahmadi and Firoozbakhsh developed a first strain gradient theory of thermoelasticity[7].

In the present work the mechanics of deformation of a nondissipative liquid filled saturated porous elastic medium with microstructure is considered. Employing the method of Mindlin[4] and Mindlin and Eshel[5], a first strain gradient theory of porous elastic continuum is derived. A linear theory is developed and the basic dynamical equations for infinitesimal motion of fluid–solid system are obtained and discussed. Wave propagation is considered and the existance of five elastic body waves, two shear waves and three dilatational waves are established.

The theory finds numerous applications in a diversity of fields, including geophysics, seismology, civil engineering, and acoustics.

2. GOVERNING EQUATIONS

In this section the principle of conservation of momentum, angular momentum and energy are employed in the derivation of governing equations of the first strain gradient theory of porous elastic continuum.

We shall use the total stress components of bulk material t_{ij} such that

$$t_{ij} = \sigma_{ij} + \delta_{ij}\sigma \tag{2.1}$$

if we consider a unit cube of bulk material, the components σ_{ij} represent the force applied to the solid part of the faces, and σ represents the force applied to the fluid part of these faces.

Let f_i and m_i be the components of force and couple, per unit area, acting on the solid part of a surface S of a body occupying a volume v; and let F_i be the components of the force acting on the fluid part of surface S; also let g_i and C_i be the components of force and couple per unit mass of bulk material and the solid part respectively, then the principle of local balance of linear and angular momentum can be expressed by

$$t_{jk,j} + \rho g_k = \rho_1 \ddot{u}_k + \rho_2 \ddot{U}_k \tag{2.2}$$

$$\mu_{ji,j} + e_{jik}\sigma_{kj} + \rho_1 C_i = 0 \tag{2.3}$$

where we have defined the stress tensor σ_{ij} and couple stress tensor μ_{ij} in solid such that

$$f_j = n_i\sigma_{ij}; \qquad m_j = n_i\mu_{ij} \tag{2.4}$$

also for the fluid part,

$$F_j = n_i\sigma\delta_{ij} \tag{2.5}$$

and we have used the divergence theorem to lead to local conservation equations. In (2.2) and (2.3) u_i and U_i, respectively, define the displacement of the solid matrix and the average fluid displacement. Vector U_i is defined such that fU_i, where f denotes the prosity, defines the volume of fluid displaced through unit areas normal to x_i direction. ρ_1 and ρ_2 are related to the mass density of the fluid, ρ_f, and the mass density of solid, ρ_s, by

$$\rho_2 = f\rho_f; \qquad \rho_1 = (1-f)\rho_s \tag{2.6}$$

and

$$\rho = \rho_1 + \rho_2 \tag{2.7}$$

represents the total mass of bulk material per unit volume. With p_f denoting the fluid pressure, we also have

$$\sigma = -fp_f. \tag{2.8}$$

If we evaluate the antisymmetric part of stress tensor t_{ij} from (2.3) and substitute it into (2.2), the result is

$$t_{(jk),j} - \tfrac{1}{2}\mu^D_{im,ij}e_{jkm} + \rho g_k - \tfrac{1}{2}\rho_1 C_{m,j}e_{jkm} = \rho_1\ddot{u}_k + \rho_2\ddot{U}_k \tag{2.9}$$

where μ^D_{ij} is the deviatoric part of μ_{ij}.

We now define the strain energy of a Porous elastic medium as the isothermal free energy of the fluid–solid system. If W denotes the strain energy per unit volume, extending the assumption of Mindlin and Eshel[5], we consider

$$W = \bar{W}(\xi, \epsilon_{ij}, \bar{K}_{ij}, \bar{\bar{K}}_{ijk}) \tag{2.10}$$

where

$$\epsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) = \epsilon_{ji} = \text{strain of solid} \tag{2.11}$$

$$\bar{K}_{ij} = \tfrac{1}{2}e_{jmk}u_{k,mi} = \text{gradient of solid rotation } (\bar{K}_{ii} = 0) \tag{2.12}$$

$$\bar{\bar{K}}_{ijk} = 1/3(u_{k,ij} + u_{i,jk} + u_{j,ki}) = \bar{\bar{K}}_{jki} = \bar{\bar{K}}_{kij} = \bar{\bar{K}}_{kji} \tag{2.13}$$

= symmetric part of second gradient of solid displacement and

$$\xi = -\nabla\cdot\mathbf{w} \tag{2.14}$$

where

$$w_i = f(U_i - u_i) \tag{2.15}$$

represents the flow of the fluid relative to the solid but measured in terms of volume per unit area of the bulk medium.

For uniform porosity we can write (2.14) as

$$\xi = f\nabla\cdot(\mathbf{u}-\mathbf{U}) \tag{2.16}$$

the variable ξ which was introduced by Biot[8, 9], is a measure of the amount of fluid which has flowed in and out of a given element attached to the solid frame, i.e. it represents the increment of fluid content. Furthermore introducing

$$p_f = \rho\frac{\partial \bar{W}}{\partial \xi} = \text{fluid pressure} \tag{2.17}$$

$$\bar{t}_{ij} = \rho\frac{\partial \bar{W}}{\partial \epsilon_{ij}} = \bar{t}_{ji} = \text{stress tensor} \tag{2.18}$$

$$\bar{\mu}_{ij} = \rho\frac{\partial \bar{W}}{\partial \bar{K}_{ij}} = \text{deviator of couple stress } (\bar{\mu}_{ii} = 0) \tag{2.19}$$

$$\bar{\bar{\mu}}_{ijk} = \rho\frac{\partial \bar{W}}{\partial \bar{\bar{K}}_{ijk}} = \bar{\bar{\mu}}_{jki} = \bar{\bar{\mu}}_{kij} = \bar{\bar{\mu}}_{kji} = \text{double stress tensor} \tag{2.20}$$

we will have

$$\rho\dot{W} = p_f\dot{\xi} + \bar{t}_{ij}\dot{\epsilon}_{ij} + \bar{\mu}_{ij}\dot{\bar{K}}_{ij} + \bar{\bar{\mu}}_{ijk}\dot{\bar{\bar{K}}}_{ijk}. \tag{2.21}$$

The kinetic energy of a unit volume of bulk material with statistical isotropy of microvelocity field can be written in the form[10]

$$T = \tfrac{1}{2}\rho\dot{u}_i\dot{u}_i + \rho_f\dot{u}_i\dot{w}_i + \tfrac{1}{2}m\dot{w}_i\dot{w}_i \tag{2.22}$$

where

$$m_{ij} = m\delta_{ij} \tag{2.23}$$

and m_{ij} is related to the components of the relative micro velocity field in the pores, v_i, by

$$\rho_f\int_V v_iv_i\,\mathrm{d}v = m_{ij}\dot{w}_i\dot{w}_j \tag{2.24}$$

and

$$v_i = a_{ij}\dot{w}_j \tag{2.25}$$

with (2.4), (2.5), (2.8), (2.22), divergence theorem and chain rule, following the same technique as [7], the principle of conservation of energy for the isothermal case is reduced to

$$\rho\dot{w} + \rho_f\ddot{u}_i\dot{w}_i + m\dot{w}_i\ddot{w}_i = -p_{f,i}\dot{w}_i - p_f\dot{w}_{i,i} + [t_{(jk)} + \bar{\bar{\mu}}_{ijk,i}]\dot{\epsilon}_{jk} + \mu^D_{ij}\dot{\bar{K}}_{ij} + \bar{\bar{\mu}}_{ijk}\dot{\bar{\bar{K}}}_{ijk}. \tag{2.26}$$

Finally inserting (2.21) in the left side of (2.26) and equating coefficients of like kinematic variables on both side of the equation, we find

$$t_{(jk)} = \bar{t}_{jk} - \bar{\bar{\mu}}_{ijk,i} \tag{2.27}$$

$$\mu^D_{ij} = \bar{\mu}_{ij} \tag{2.28}$$

$$-p_{f,j} = \rho_f\ddot{u}_j + m\ddot{w}_j \tag{2.29}$$

upon substituting (2.27) and (2.28) into (2.9) we obtain the basic equations of motion of the generalized porous elastic body

$$\bar{t}_{jk,j} - \tfrac{1}{2}e_{jkm}\bar{\mu}_{im,ij} - \bar{\bar{\mu}}_{ijk,ij} + \rho g_k - \tfrac{1}{2}\rho_1 e_{jkm}C_{m,j} = \rho_1\ddot{u}_k + \rho_2\ddot{U}_k. \tag{2.30}$$

This coupled with eqn (2.29) gives the general equations of the first strain gradient theory of porous elastic material.

3. LINEAR CONSTITUTIVE EQUATIONS

For a homogeneous centrosymmetric isotropic, isothermal medium, the most general form of a positive definite strain energy (2.10) which leads to linear constitutive equations is

$$\rho_0 \bar{W} = W_0 - \rho_0 p_{f0} \xi - \frac{\rho_0 \beta}{2\xi_0} \xi^2 - \gamma \epsilon_{ii} \xi + \frac{1}{2} \lambda \epsilon_{ii} \epsilon_{jj} + \mu \epsilon_{ij} \epsilon_{ij}$$
$$+ 2\bar{d}_1 \bar{K}_{ij} \bar{K}_{ij} + 2\bar{d}_2 \bar{K}_{ij} \bar{K}_{ji} + 3/2 \bar{a}_1 \bar{\bar{K}}_{iij} \bar{\bar{K}}_{kkj} + \bar{a}_2 \bar{\bar{K}}_{ijk} \bar{\bar{K}}_{ijk} + \bar{f} e_{ijk} \bar{K}_{ij} \bar{\bar{K}}_{kmm}. \quad (3.1)$$

Now using (2.17)–(2.20) the following constitutive equations may be easily obtained

$$p_f = -\rho_0 p_{f0} - \rho \beta \frac{\xi}{\xi_0} - \gamma \epsilon_{ii} \quad (3.2)$$

$$\bar{t}_{ij} = +(\lambda \epsilon_{kk} - \gamma \xi) \delta_{ij} + 2\mu \epsilon_{ij} \quad (3.3)$$

$$\bar{\mu}_{ij} = 4\bar{d}_1 \bar{K}_{pq} + 4 d_2 \bar{K}_{qp} + \bar{f} e_{pqi} \bar{\bar{K}}_{ijj} \quad (3.4)$$

$$\bar{\bar{\mu}}_{pqr} = \bar{a}_1 (\bar{\bar{K}}_{iir} \delta_{pq} + \bar{\bar{K}}_{iip} \delta_{qr} + \bar{\bar{K}}_{iiq} \delta_{rp}) + 2\bar{a}_2 \bar{\bar{K}}_{pqr}$$
$$+ 1/3 \bar{f} \bar{K}_{ij} (\delta_{pq} e_{ijr} + \delta_{qr} e_{ijp} + \delta_{rp} e_{ijq}) \quad (3.5)$$

when (3.3)–(3.5) are inserted in (2.30) and ϵ_{ij}, $\bar{K}_{ij}$ and $\bar{\bar{K}}_{ijk}$ are replaced by their expressions in terms of u_i for constant values of the parameters we find,

$$(\lambda + 2\mu)(1 - l_1^2 \nabla^2) \nabla\nabla \cdot \mathbf{u} - \mu (1 - l_2^2 \nabla^2) \nabla \times \nabla \times \mathbf{u} + \rho \mathbf{g} + \tfrac{1}{2} \rho_1 \nabla \times \mathbf{C} - \gamma \nabla \xi = \rho_1 \ddot{\mathbf{u}} + \rho_2 \ddot{U} \quad (3.6)$$

where

$$l_1^2 = (3\bar{a}_1 + 2\bar{a}_2)/\lambda + 2\mu \quad (3.7)$$

$$l_2^2 = (3\bar{d}_1 + \bar{a}_1 + 2\bar{a}_2 - \bar{f})/3\mu \quad (3.8)$$

the necessary and sufficient conditions for positive definitness of strain energy W are [5, 10]

$$\mu \geq 0, \quad \lambda + 2/3\mu \geq 0, \quad -\bar{d}_1 < \bar{d}_2 < \bar{d}_1$$
$$\bar{a}_2 > 0, \quad 5\bar{a}_1 + 2\bar{a}_2 > 0, \quad 5\bar{f}^2 < 6(\bar{d}_1 - \bar{d}_2)(5\bar{a}_1 + 2\bar{a}_2)$$
$$\beta \geq 0, \quad \gamma \geq 0$$

which imply

$$l_1^2 > 0, \quad l_2^2 > 0 \quad (3.9)$$

furthermore, when we substitute expression (3.2) for p_f eqn (2.29) become,

$$\frac{\rho \beta}{\xi_0} \nabla \xi + \gamma \nabla\nabla \cdot \mathbf{u} = \rho_f \ddot{\mathbf{u}} + m \ddot{\mathbf{w}}. \quad (3.10)$$

Equations (3.6) and (3.10) are the generalized form of equations developed by Biot[10], in which the coefficients are given by following expressions:

$$M = -\frac{\beta \rho}{\xi_0}; \quad \alpha = \frac{\gamma}{M} = -\frac{\gamma \xi_0}{\beta \rho}. \quad (3.11)$$

Note that in the infinitesimal motion theory $\rho = \rho_0$.

If we multiply eqn (3.10) by f and then subtract it from eqn (3.6) after some rearrangements we obtain

$$[(\lambda + 2\mu)(1 - l_1^2 \nabla^2) - (\gamma f + Q)] \nabla\nabla \cdot \mathbf{u} - \mu (1 - l_2^2 \nabla^2) \nabla \times \nabla \times \mathbf{u} + Q \nabla\nabla \cdot \mathbf{U} + \rho \mathbf{g} + \tfrac{1}{2} \rho_1 \nabla \times \mathbf{C}$$
$$= \rho_{11} \ddot{\mathbf{u}} + \rho_{12} \ddot{\mathbf{U}} \quad (3.12)$$

$$Q \nabla\nabla \cdot \mathbf{u} + R \nabla\nabla \cdot \mathbf{U} = \rho_{12} \ddot{\mathbf{u}} + \rho_{22} \ddot{\mathbf{U}}. \quad (3.13)$$

Equations (3.12) and (3.13) are the basic equations of the first strain gradient theory of porous elastic material. In (3.12) and (3.13) we have defined the relations

$$\begin{aligned}
\rho_{12} &= f\rho_f - mf^2 \\
\rho_{11} &= \rho - 2f\rho_f + mf^2 \\
\rho_{22} &= mf^2 \\
Q &= f\left(\frac{\rho\beta}{\xi_0}f + \gamma\right) \\
R &= -f^2\frac{\rho\beta}{\xi_0}
\end{aligned} \tag{3.14}$$

in the absence of body force and neglecting couple stresses relations (3.12) and (3.13) reduces to those given in [9, 10].

4. WAVE PROPAGATION

In order to study eqns (3.12) and (3.13) we utilize the Helmholtz resolution of the displacement vector,

$$\begin{aligned}
\mathbf{u} &= \nabla\phi + \nabla\times\mathbf{H}; \quad \nabla\cdot\mathbf{H} = 0 \\
\mathbf{U} &= \nabla\psi + \nabla\times\mathbf{G}; \quad \nabla\cdot\mathbf{G} = 0
\end{aligned} \tag{4.1}$$

wherein ϕ and ψ are the scalar potentials and $\mathbf{H}$ and $\mathbf{G}$ are the vector potentials. With (4.1), eqns (3.12) and (3.13) decomposes into four equations in terms of potentials

$$\begin{aligned}
[(\lambda+2\mu)(1-l_1^2\nabla^2)-(\gamma f+Q)]\nabla^2\phi + Q\nabla^2\psi + \rho g_A &= \rho_{11}\ddot{\phi} + \rho_{12}\ddot{\psi} \\
Q\nabla^2\phi + R\nabla^2\psi &= \rho_{12}\ddot{\phi} + \rho_{22}\ddot{\psi}
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
\mu(1-l_2^2\nabla^2)\nabla^2\mathbf{H} + \rho\mathbf{g}_B + \tfrac{1}{2}\rho_1\mathbf{C} &= \rho_{11}\ddot{\mathbf{H}} + \rho_{12}\ddot{\mathbf{G}} \\
0 &= \rho_{12}\ddot{\mathbf{H}} + \rho_{22}\ddot{\mathbf{G}}
\end{aligned} \tag{4.3}$$

where we have assumed

$$\mathbf{g} = \nabla g_A + \nabla\times\mathbf{g}_B; \qquad \nabla\cdot\mathbf{g}_B = 0. \tag{4.4}$$

It is interesting to note that the field equations for the vector potentials are decoupled from those of the scalar potentials. Equations (4.2) and (4.3) are the governing equations for the propagation of dilatational and distortional waves, respectively.

It is further assumed that the scalar potentials, ϕ and ψ, have a harmonic time variation, i.e.

$$\begin{aligned}
\phi(x_i, t) &= \bar{\Phi}(x_i)\exp(i\omega t) \\
\psi(x_i, t) &= \bar{\Psi}(x_i)\exp(i\omega t)
\end{aligned} \tag{4.5}$$

where ω is the frequency. For vanishing body force, insertion of (4.5) in (4.2), upon elimination of $\nabla^2\bar{\Psi}$ yield

$$\bar{\Psi} = (\rho_{22}Q - \rho_{12}R)^{-1}\left[\frac{-RP}{\omega^2}l_1^2\nabla^4\bar{\Phi} + \frac{A}{\omega^2}\nabla^2\bar{\Phi} + (\rho_{11}R - \rho_{12}Q)\bar{\Phi}\right] \tag{4.6}$$

and

$$-RPl_1^2\nabla^6\bar{\Phi} + [A - \rho_{22}\omega^2(\lambda+2\mu)l_1^2]\nabla^4\bar{\Phi} + B\omega^2\nabla^2\bar{\Phi} + C\omega^4\bar{\Phi} = 0 \tag{4.7}$$

where,

$$P = \lambda + 2\mu; \quad A = PR - Q^2 - R(\gamma f + Q)$$
$$C = \rho_{11}\rho_{22} - \rho_{12}^2; \quad B = \rho_{11}R + \rho_{22}P - 2\rho_{12}Q - \rho_{22}(\gamma f + Q). \tag{4.8}$$

Similarly, we assume

$$\mathbf{H}(x_j, t) = \bar{\mathbf{H}}(x_j)\exp(i\omega t)$$
$$\mathbf{G}(x_j, t) = \bar{\mathbf{G}}(x_j)\exp(i\omega t). \tag{4.9}$$

For vanishing body force and body couple, insertion of (4.9) in (4.3) yield

$$\bar{\mathbf{G}} = -\frac{\rho_{12}}{\rho_{22}}\bar{\mathbf{H}} \tag{4.10}$$

and

$$(1 - l_2^2\nabla^2)\nabla^2\bar{\mathbf{H}} + \delta^2\bar{\mathbf{H}} = 0 \tag{4.11}$$

where

$$\delta^2 = \lambda^2\omega^2; \quad \lambda^2 = \frac{C}{\mu\rho_{22}}. \tag{4.12}$$

For $l_1 = l_2 = 0$ eqns (4.6), (4.7), (4.10) and (4.11) reduces to the classical wave equations of porous elastic media (see for instance [11]). In an unbounded medium, studies of eqns (4.7) and (4.11) will reveal the existance of three dilatational waves and two shear waves, respectively. This is incontrast to the previous classical theory in which there existed only two dilatational waves and one shear wave[9, 11]. In the absence of dissipation these waves are elastic in nature, the propagation being at constant velocity and with undiminished amplitude.

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